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LETTER TO THE EDITOR

$\mathcal{PT}$-invariant point interactions in one dimension

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Abstract

By using Wu and Yu's pseudo-potential, we construct point interactions in one dimension that are complex but conform to space–time reflection ($\mathcal{PT}$) invariance. The resulting point interactions are equivalent to those obtained by Albeverio, Fei and Kurasov as self-adjoint extensions of the kinetic energy operator.

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There are point interactions in one-dimensional quantum mechanics in the form of the pseudo-potential proposed by Wu and Yu (WY) [1]. On the other hand there are point interactions that can be interpreted as self-adjoint extensions (SAEs) of the kinetic energy (KE) operator $-(\hbar^2/2m)\nabla^2$. For the SAEs, see, for example [2–5]. We recently pointed out that there is a one-to-one correspondence between the point interactions of these two forms [6]. WY's pseudo-potential is a convenient device which enables us to obtain SAEs of the KE operator. When time-reversal invariance is imposed, the point interactions can have three parameters.

Recently there has been a surge of interest in Hamiltonians that are complex but pseudo-Hermitian, i.e., invariant under space–time reflection ($\mathcal{PT}$ symmetry). Here $\mathcal{P}$ and $\mathcal{T}$ respectively stand for space reflection ($x \to -x$) and time-reversal operations ($t \to -t$). In particular, Albeverio, Fei and Kurasov extensively examined $\mathcal{PT}$-invariant point interactions [7]. Other references regarding the $\mathcal{PT}$-invariant Hamiltonian can be traced through [7–10]. The purpose of this letter is to present a $\mathcal{PT}$-invariant version of the results obtained in [6]. For notational brevity we take units such that $\hbar^2/(2m) = 1$ where m is the mass of the particle of the system under consideration.

We consider the time-independent Schrödinger equation in one dimension in the usual notation

$$-\psi''(x) + \int_{-\infty}^{\infty} V(x, x')\psi(x')\,\mathrm{d}x' = E\psi(x), \tag{1}$$

where $\psi''(x) = \mathrm{d}^2\psi(x)/\mathrm{d}x^2$. For potential $V(x, x')$, following WY, we assume the pseudo-potential of the form of

$$V(x, x') = g_1 v_1(x, x') + g_2 v_2(x, x') + g_3 v_3(x, x'), \tag{2}$$

where

$$\begin{aligned} &v_1(x, x') = \delta(x)\delta(x'), \qquad v_2(x, x') = \delta'_p(x)\delta(x') + \delta(x)\delta'_p(x'), \\ &v_3(x, x') = \delta'_p(x)\delta'_p(x'). \end{aligned} \tag{3}$$

The $\delta'_p(x)$ is defined by

$$\delta'_p(x)\psi(x) = \delta'(x)\tilde{\psi}(x), \tag{4}$$

where $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$ and

$$\tilde{\psi}(x) = \begin{cases} \psi(x) - \frac{1}{2}(\psi_+ - \psi_-) & \text{for } x > 0, \\ \psi(x) + \frac{1}{2}(\psi_+ - \psi_-) & \text{for } x < 0. \end{cases} \tag{5}$$

Subscript + (−) refers to the boundary value for $x \to +0$ ($x \to -0$), e.g., $\psi_+ = \psi(+0)$. Note that $\tilde{\psi}(x)$ is continuous at $x = 0$ and $\tilde{\psi}(0) = (1/2)(\psi_+ + \psi_-)$. It is understood that $\psi(x)$ is generally discontinuous at $x = 0$, i.e., $\psi_+ \neq \psi_-$. Actually the $\tilde{\psi}(x)$ defined above is different from WY's $\tilde{\psi}(x)$ by an additive constant. (See equation (31) of [6].) The $\delta'_p(x)$ is anti-symmetric, i.e., $\delta'_p(-x) = -\delta'_p(x)$. Potential $V(x, x')$ represents a point interaction at the origin.

For the strength parameters g_i ($i = 1, 2, 3$) of (2) they were all assumed to be real before so that $V(x, x')$ of (2) is Hermitian [6]. *This time, however, we assume that g_2 is pure imaginary, i.e., $g_2^* = -g_2$, while g_1 and g_3 remain as real parameters.* Recall that

$$v_1(x, x') = v_1(-x, -x'), \qquad v_2(x, x') = -v_2(-x, -x'), \qquad v_3(x, x') = v_3(-x, -x'). \tag{6}$$

With the pure imaginary g_2, $V(x, x')$ is $\mathcal{PT}$-invariant.

We are interested in the behaviour of the wavefunction around the origin. It is understood that $\psi(x)$ is twice differentiable except at $x = 0$ but $\psi(x)$ and $\psi'(x) = \mathrm{d}\psi(x)/\mathrm{d}x$ are discontinuous at $x = 0$ in general. The boundary condition on the wavefunction at $x = 0$ can be expressed in the following form,

$$\begin{pmatrix} \psi'_+ \\ \psi_+ \end{pmatrix} = U \begin{pmatrix} \psi'_- \\ \psi_- \end{pmatrix}, \qquad U = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \tag{7}$$

where α, β, γ, δ and θ are all constants. The θ is real. (We do not consider the cases in which the two half-spaces of $x > 0$ and $x < 0$ are disjoint.) Equations (1) and (2) lead to

$$U = \frac{\mathrm{e}^{\mathrm{i}\theta}}{4\Delta} \begin{pmatrix} (2 - g_2)^2 - g_1 g_3 & 4g_1 \\ -4g_3 & (2 + g_2)^2 - g_1 g_3 \end{pmatrix}, \tag{8}$$

$$\Delta = \tfrac{1}{4}[(2 + g_2)(2 - g_2) + g_1 g_3]. \tag{9}$$

Note that α and γ ($= \alpha^*$) are complex while β and δ are real. (If g_2 is real, α and γ are also real.) It is understood that $\Delta \neq 0$. The α, etc, satisfy the condition

$$\alpha\gamma - \beta\delta = 1. \tag{10}$$

Hence among α, β, γ and δ there are only three real independent parameters. This is as it should be because we started with three parameters, g_1, g_2 and g_3. The parametrization of the

boundary condition presented above can be rewritten into the form of (4) of Albeverio *et al* [7]. Our parameters are related to those of [7] by

$$\alpha = \gamma^* = \sqrt{1+bc}\,\mathrm{e}^{-\mathrm{i}\phi}, \qquad \beta = c \quad \delta = b. \tag{11}$$

Parameter θ is unimportant for the same reason as pointed out in [11]. In the following we choose θ as

$$\mathrm{e}^{\mathrm{i}\theta} = -1. \tag{12}$$

This is to conform to the notation that was used in some of the earlier papers [2, 3, 5]. (If we choose $\mathrm{e}^{\mathrm{i}\theta} = 1$ instead, the signs of α, etc, in the following formulae are all reversed.)

Let us examine the transmission–reflection problem. If a wave of a specified wavelength is incident from the left, the wavefunction can be written as [12]

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_{\mathrm{L}}\,\mathrm{e}^{-\mathrm{i}kx} & \text{for } x < 0 \\ T_{\mathrm{L}}\,\mathrm{e}^{\mathrm{i}kx} & \text{for } x > 0, \end{cases} \tag{13}$$

where $k > 0$ is related to the energy by $E = k^2$. The wavefunction of the case in which the wave is incident from the right can be written in a similar manner, with coefficients T_{R} and R_{R}. The S-matrix is a 2×2 matrix. It is related to T and R by,

$$S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix} = \begin{pmatrix} T_{\mathrm{L}} & R_{\mathrm{R}} \\ R_{\mathrm{L}} & T_{\mathrm{R}} \end{pmatrix}. \tag{14}$$

The '$\pm$' of S_{++}, etc, unlike the '$\pm$' of $\psi_\pm$ that we introduced in (5), refer to the direction of the wave propagation. By solving the Schrödinger equation (1) with pseudo-potential (2) we obtain

$$S = \left[\mathrm{i}g_3k + \frac{1}{2}\left(4 - g_1g_3 + g_2^2\right) + \mathrm{i}g_1k^{-1}\right]^{-1} \begin{pmatrix} \frac{1}{2}\left(4 + g_1g_3 - g_2^2\right) & \mathrm{i}g_3k + 2g_2 - \mathrm{i}g_1k^{-1} \\ \mathrm{i}g_3k - 2g_2 - \mathrm{i}g_1k^{-1} & \frac{1}{2}\left(4 + g_1g_3 - g_2^2\right) \end{pmatrix} \tag{15}$$

$$= [-\beta + k^2\delta + \mathrm{i}k(\alpha+\gamma)]^{-1} \begin{pmatrix} -2\mathrm{i}k & \beta + k^2\delta - \mathrm{i}k(\alpha-\gamma) \\ \beta + k^2\delta + \mathrm{i}k(\alpha-\gamma) & -2\mathrm{i}k \end{pmatrix}. \tag{16}$$

Note that $T_{\mathrm{L}} = T_{\mathrm{R}}$ and $R_{\mathrm{L}} \neq R_{\mathrm{R}}$.

The S-matrix obtained above is not unitary. Consequently the probability (defined in a conventional way) is not conserved. This can be seen from $|T|^2 + |R_{\mathrm{L}}|^2 \neq |T|^2 + |R_{\mathrm{R}}|^2$ where $T = T_{\mathrm{L}} = T_{\mathrm{R}}$. Here we should mention the following. Deb *et al* pointed out that actually any $\mathcal{PT}$-invariant interaction leads to $T_{\mathrm{L}} = T_{\mathrm{R}}$ [13]. They also worked out the T and R explicitly for certain models with $\mathcal{PT}$-invariant interactions and illustrated the probability non-conservation. The usual SAEs of the KE operator can be obtained by requiring that the conventional probability current $-\mathrm{i}(\psi^*\psi' - \psi'^*\psi)$ be continuous across the origin [5]. Boundary condition (7) with complex α and γ, however, is not compatible with this continuity. On the other hand (7) guarantees the continuity of $-\mathrm{i}[(\mathcal{PT}\psi)\psi' - (\mathcal{PT}\psi)'\psi]$ across the origin where $\mathcal{PT}\psi(x) = \psi^*(-x)$ [7].

The case that we have considered is the one-channel case in which the wavefunction of the particle has only one component. The analysis can be extended to the two-channel case. Then the pseudo-potential obtains ten parameters, which we denoted by f_i, g_i, h_i and η where $i = 1$, 2 or 3 in [6]. The two-channel version of the $\mathcal{PT}$-invariant point interactions can be obtained by changing f_2, g_2 and h_2 from real to pure imaginary.

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